GIRRAWEEN HIGH SCHOOL

MATHEMATICS

Year 12 Extension 1, Task 3 **May 2006**

Instructions: **Time: 90 minutes**

* Start each question on a separate page.
* Show all necessary working.
* Marks may be deducted for careless or badly arranged work.

Question 1 (16 marks)

(a) Evaluate the following angles as an exact answer in terms of π. 3

(i) $\sin^{-1}(-1)$ (ii) $\cos^{-1}(-\frac{\sqrt{3}}{2})$ (iii) $\tan^{-1}(\frac{1}{\sqrt{3}})$

(b) Evaluate without using calculator. 8

(i) $\sin(2\sin^{-1}\frac{4}{5})$ (ii) $\cos[\sin^{-1}\frac{3}{4}+\cos^{-1}\frac{5}{13}]$

(c) For the function $y=\sin^{-1}(3x)$

(i) State the domain and range. 3

(ii) Sketch the graph. 2

Question 2 (22 marks)

(a) Find the general solution in radians of the following:

(i) $\cos\theta=\frac{\sqrt{3}}{2}$ (ii) $\tan\theta=\sqrt{3}$ 4

(b) Differentiate: 14

(i) $y=\sin^{-1}3x$ (ii) $y=e^{x}\sin^{-1}x$

(iii) $y=\cos^{-1}\left(\frac{x}{4}\right)$ (iv) $y=(\tan^{-1}x)^{6}$

(v) $e^{\cos^{-1}(5x+3)}$

(c)Find the equation of the normal to the curve $y=\tan^{-1}5x$

at $(\frac{1}{5},\frac{\pi}{4})$ 4

Question 3 (23 marks)

(a) Find the following indefinite integrals: 10

(i) $\int \frac{dx}{\sqrt{6-x^2}}$ (ii) $\int \frac{dx}{1+4x^2}$

(iii) $\int \frac{dx}{1+(x+4)^2}$ (iv) $\int \frac{dx}{4(x-3)^2+9}$

(b) Evaluate: 6

(i) $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$ (ii) $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$

(c) (i) Find $\frac{d}{dx}(x\sin^{-1}x+\sqrt{1-x^2})$ 7

(ii) Hence or otherwise find the exact area bounded by the curve $y=\sin^{-1}x$, the x axis and the lines $x=\frac{1}{2}$ and $x=\frac{\sqrt{3}}{2}$.

Question 4 (19 marks)

(a) For each of the following functions, sketch the graph and state whether the inverse function exists or not. If not, find the largest domain containing the origin for which an inverse function exists.

(i) $f(x)=(x-4)^2$ (ii) $f(x)=e^x+1$ 5

(b) Given $f(x)=x^2-2x,\ x\geq 1$

(i) State the domain and range of both $f(x)$ and $f^{-1}(x)$. 4

(ii) Find $f^{-1}(x)$. 4

(iii) Graph $f(x)$ and $f^{-1}(x)$. 2

(c) Verify that f and g are inverse functions by showing that

$f(g(x))=g(f(x))=x$ where $f(x)=3x-1$ and $g(x)=\frac{x+1}{3}$. 4

Question 5 (30 marks)

(a) In each of the following, use the given substitution to find the primitive functions.

(i) $\int x\sqrt{2+x^2}\,dx$, $u=2+x^2$

(ii) $\int \frac{\ln 2x}{x}\,dx$, $u=\ln 2x$

(iii) $\int \frac{e^x}{1+e^{2x}}\,dx$, $u=e^x$

(iv) $\int \frac{(\tan^{-1}x)^2}{1+x^2}\,dx$, $u=\tan^{-1}x$ 12

(b) Evaluate the following definite integrals using the given substitution. 12

(i) $\int_2^{10} \frac{x}{\sqrt{x-1}}\,dx$, $u^2=x-1$

(ii) $\int_1^4 \frac{dx}{\sqrt{x}\,(1+x)}$, $u=\sqrt{x}$

(iii) $\int_0^{\frac{\pi}{2}} \tan^2\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx$, $u=\tan\frac{x}{2}$

(c) (i) If $x=4\sin\theta$, show that $\cos\theta=\frac{\sqrt{16-x^2}}{4}$ 6

(ii) By substituting $x=4\sin\theta$, show that

$$\int \frac{x^2\,dx}{\sqrt{16-x^2}} = 8\sin^{-1}\left(\frac{x}{4}\right)-\frac{1}{2}x\sqrt{16-x^2}+C$$

Year 12 3 Unit Task 3, 2006 - Solutions

Question 1 (16 marks)

(a) (i) $\sin^{-1}(-1) = \frac{-\pi}{2}$ 1

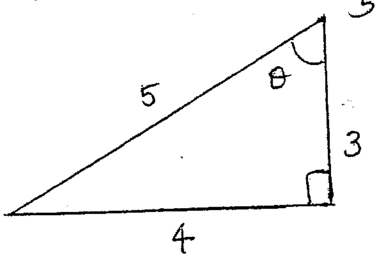
(ii) $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{5\pi}{6}$ 1

(iii) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ 1

(b) (i) $\sin\left(2\sin^{-1}\frac{4}{5}\right)$

Let $\theta = \sin^{-1}\frac{4}{5}$

Then $\sin\theta = \frac{4}{5}$



4

$\sin\left(2\sin^{-1}\frac{4}{5}\right)$

$= \sin(2\theta)$

$= 2\sin\theta\cos\theta$

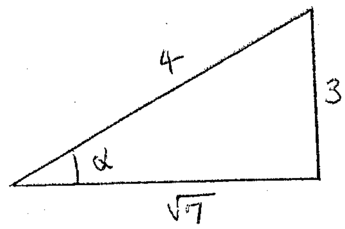
$= 2 \times \frac{4}{5} \times \frac{3}{5}$

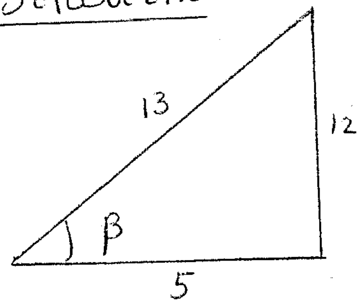
$= \frac{24}{25}$

(ii) $\cos\left[\sin^{-1}\frac{3}{4} + \cos^{-1}\frac{5}{13}\right]$

Let $\alpha = \sin^{-1}\frac{3}{4}$ and $\beta = \cos^{-1}\frac{5}{13}$

$\sin\alpha = \frac{3}{4}$ $\quad\cos\beta = \frac{5}{13}$





$\cos\alpha = \frac{\sqrt{7}}{4}$ $\quad\sin\beta = \frac{12}{13}$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{\sqrt{7}}{4} \times \frac{5}{13} - \frac{3}{4} \times \frac{12}{13}$ 4

$= \frac{5\sqrt{7} - 36}{52}$

(c) $y = \sin^{-1}(3x)$

(i) **Domain**

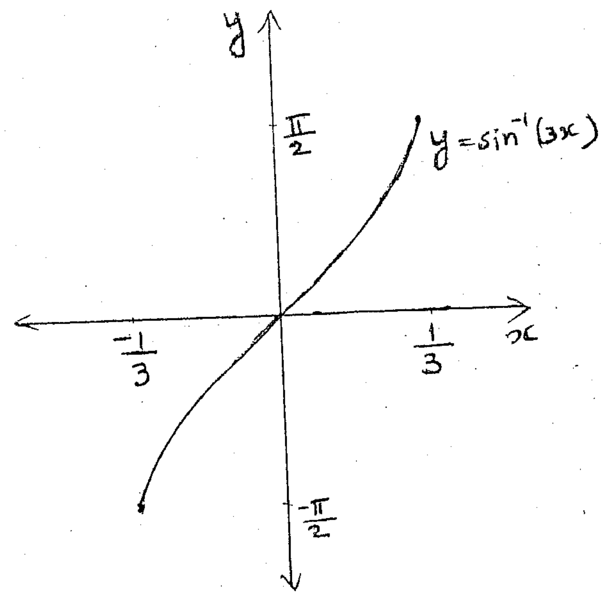
$-1 \le 3x \le 1$ 2

$\frac{-1}{3} \le x \le \frac{1}{3}$

Range

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ 1

(ii)



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Question 2 (22 marks)

(a) (i) $\cos\theta = \frac{\sqrt{3}}{2}$

$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$ (2)

General solution is

$\theta = 2n\pi \pm \frac{\pi}{6}$, n an integer

(ii) $\tan\theta = \sqrt{3}$

$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ (2)

General solution is

$\theta = n\pi + \frac{\pi}{3}$ where n is any integer.

(b) (i) $y = \sin^{-1} 3x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-9x^2}} \times 3$$

$$= \frac{3}{\sqrt{1-9x^2}}$$ (2)

(ii) $y = e^x \sin^{-1} x$

$$\frac{dy}{dx} = e^x \times \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x \times e^x$$

$$= \frac{e^x}{\sqrt{1-x^2}} + e^x \sin^{-1} x$$ (2)

(iii) $y = \cos^{-1}\left(\frac{x}{4}\right)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\frac{x^2}{16}}} \times \frac{1}{4}$$

$$= \frac{-1}{\sqrt{\frac{16-x^2}{16}}} \times \frac{1}{4}$$ (3)

$$= \frac{-4}{\sqrt{16-x^2}} \times \frac{1}{4} = \frac{-1}{\sqrt{16-x^2}}$$

(iv) $y = (\tan^{-1}x)^6$

$$\frac{dy}{dx} = 6(\tan^{-1}x)^5 \times \frac{1}{1+x^2}$$

$$= \frac{6(\tan^{-1}x)^5}{1+x^2}$$ (3)

(v) $y = e^{\cos^{-1}(5x+3)}$

$$\frac{dy}{dx} = e^{\cos^{-1}(5x+3)} \times \frac{-1}{\sqrt{1-(5x+3)^2}} \times 5$$

$$= \frac{-5e^{\cos^{-1}(5x+3)}}{\sqrt{1-(5x+3)^2}}$$

(4)

(c) $y = \tan^{-1} 5x$

$$\frac{dy}{dx} = \frac{1}{1+25x^2} \times 5 = \frac{5}{1+25x^2}$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{5}} = \frac{5}{1+25\times\frac{1}{25}}$$

$$= \frac{5}{2}$$

Gradient of the normal

$$= -\frac{2}{5}$$

Equation of normal

$$y - \frac{\pi}{4} = \frac{-2}{5}\left(x - \frac{1}{5}\right) \quad (4)$$

$$y - \frac{\pi}{4} = \frac{-2x}{5} + \frac{2}{25}$$

$$500y - 125\pi = -200x + 40$$

$$200x + 500y - 125\pi - 40 = 0$$

$$50x + 100y - 25\pi - 8 = 0$$

Question 3 (23 marks) (2)

(a) (i) $\displaystyle\int \frac{dx}{\sqrt{6-x^2}} = \sin^{-1}\left(\frac{x}{\sqrt{6}}\right) + C$

(ii) $\displaystyle\int \frac{dx}{1+4x^2} = \int \frac{dx}{4\left(\frac{1}{4}+x^2\right)}$

$$= \frac{1}{4}\int \frac{dx}{\left(\frac{1}{2}\right)^2 + x^2} \quad (3)$$

$$= \frac{1}{4} \times \frac{1}{\frac{1}{2}} \tan^{-1}\left(\frac{x}{\frac{1}{2}}\right) + C$$

$$= \frac{1}{2}\tan^{-1}(2x) + C$$

(2)

(iii) $\displaystyle\int \frac{dx}{1+(x+4)^2} = \tan^{-1}(x+4) + C$

(iv) $\displaystyle\int \frac{dx}{4(x-3)^2+9} = \int \frac{dx}{4\left((x-3)^2 + \frac{9}{4}\right)}$

$$= \frac{1}{4}\int \frac{dx}{(x-3)^2 + \left(\frac{3}{2}\right)^2} \quad (3)$$

$$= \frac{1}{4} \times \frac{1}{\frac{3}{2}} \tan^{-1}\left(\frac{x-3}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{2x-6}{3}\right) + C$$

(b) (i) $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1$

$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) \quad (3)$$

$$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

(ii) $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$

$$= \tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6} \quad (3)$$

(c)(i) $\frac{d}{dx}\left(x\sin^{-1}x + \sqrt{1-x^2}\right)$

$$= x \times \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x \times 1 + \frac{1}{2\sqrt{1-x^2}} \times -2x$$

(3)

$$= \sin^{-1}x$$

(ii) $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sin^{-1}x\,dx = \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$

$$= \left(\frac{\sqrt{3}}{2}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sqrt{1-\frac{3}{4}}\right) - \left(\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) + \sqrt{1-\frac{1}{4}}\right)$$

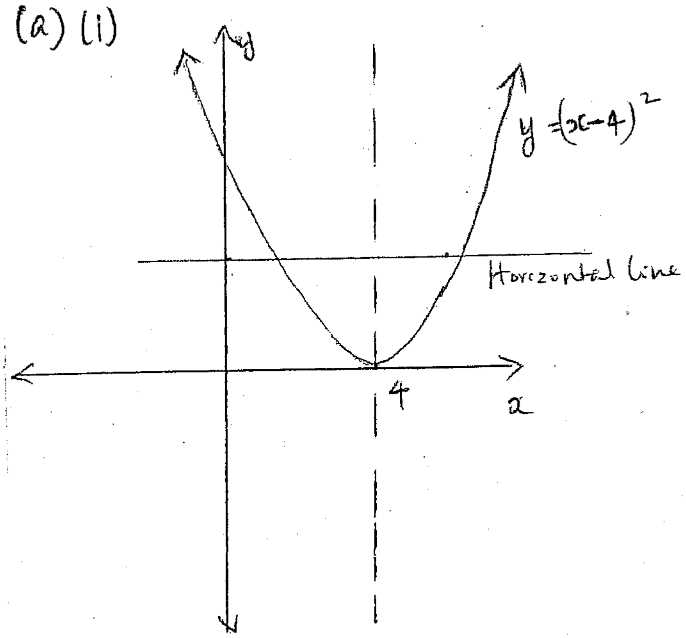
$$= \frac{\pi\sqrt{3}}{6} + \frac{1}{2} - \frac{\pi}{12} - \frac{\sqrt{3}}{2}$$

(4)

$$= \frac{\pi(2\sqrt{3}-1) + 6(1-\sqrt{3})}{12}$$

Question 4 (19 marks)

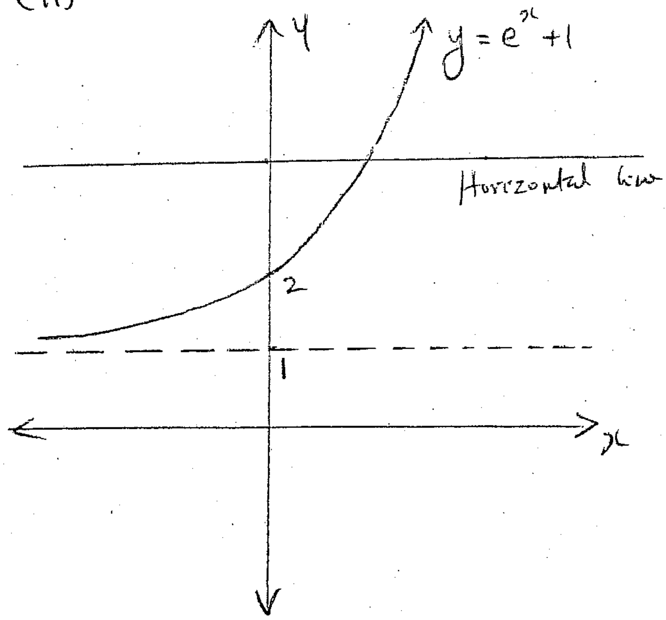
(a) (i)



Inverse does not exist for all real x. Inverse exists for $x \le 4$

(3)

(ii)



Inverse exists for all real x.

(2)

(b) $f(x) = x^2 - 2x$, $x \geq 1$

(i) axis of symmetry

$$x = \frac{-(-2)}{2} = 1$$

Vertex $(1, -1)$

D_f : $x \geq 1$ 1

R_f : $y \geq -1$ 1

$D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$

$D_{f^{-1}}$: $x \geq -1$ 1

$R_{f^{-1}}$: $y \geq 1$ 1

(ii) $f : y = x^2 - 2x$

To find f^{-1}

$$x = y^2 - 2y$$

$$x = y^2 - 2y + 1 - 1$$

$$x = (y-1)^2 - 1$$

$$x + 1 = (y-1)^2$$

$$y - 1 = \pm\sqrt{x+1}$$

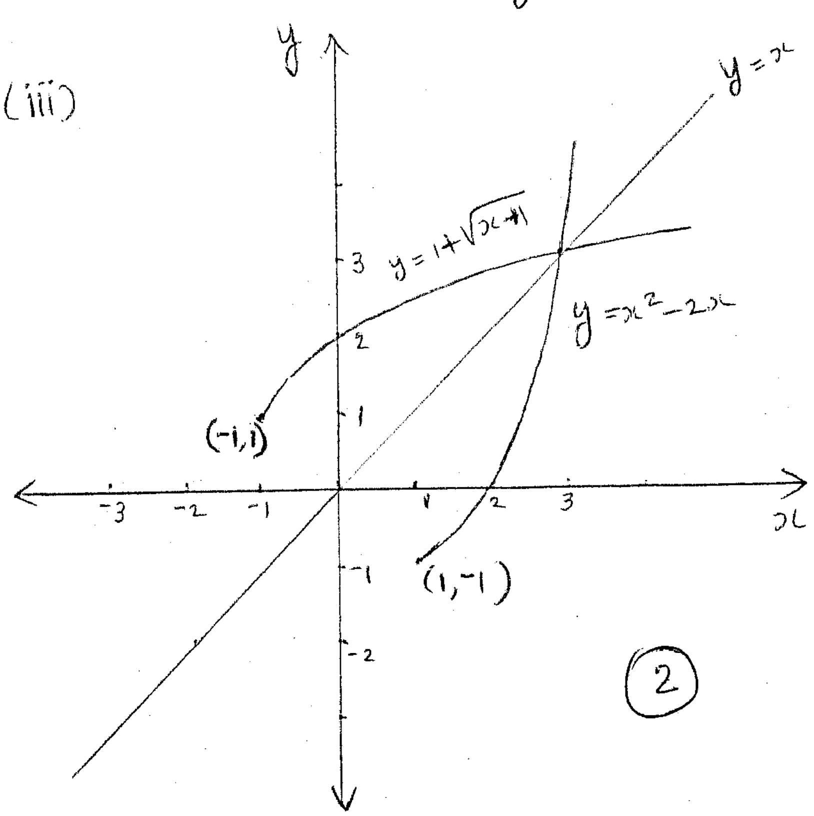
$$y = 1 \pm \sqrt{x+1}$$

Since $R_{f^{-1}}$: $y \geq 1$ 4

$$y = 1 + \sqrt{x+1}$$

f^{-1} : $y = 1 + \sqrt{x+1}$

(iii)



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(c) $f(x) = 3x - 1$ $g(x) = \dfrac{x+1}{3}$

$$f(g(x)) = f\left(\frac{x+1}{3}\right)$$

$$= 3\left(\frac{x+1}{3}\right) - 1$$

$$= x + 1 - 1 = x$$

$$g(f(x)) = g(3x-1)$$

$$= \frac{3x-1+1}{3}$$

$$= \frac{3x}{3}$$

$$= x$$

4

Since $f(g(x)) = g(f(x)) = x$,
f and g are inverse functions.

Question 5 (30 marks)

(a) (i) $\int x\sqrt{2+x^2}\,dx$

Let $u = 2+x^2$

$\frac{du}{dx} = 2x$; $2x\,dx = du$

$x\,dx = \frac{du}{2}$

$\int \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int u^{\frac{1}{2}}\,du$

$= \frac{1}{2}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{1}{2}\,u^{\frac{3}{2}} \times \frac{2}{3} + c$ (3)

$= \frac{1}{3}\,u^{\frac{3}{2}} + C$

$= \frac{1}{3}(2+x^2)^{\frac{3}{2}} + C$

(ii) $\int \frac{\ln 2x}{x}\,dx$

Let $u = \ln 2x$

$\frac{du}{dx} = \frac{1}{2x} \times 2 = \frac{1}{x}$

$du = \frac{1}{x}\,dx$ (3)

$\int u\,du = \frac{u^2}{2} + C$

$= \frac{(\log_e 2x)^2}{2} + C$

(iii) $\int \frac{e^x\,dx}{1+e^{2x}}$ Let $u = e^x$

$\frac{du}{dx} = e^x$; $e^x\,dx = du$

$\int \frac{du}{1+u^2} = \tan^{-1} u + C$ (3)

$= \tan^{-1}(e^x) + C$

(iv) $\int \frac{(\tan^{-1}x)^2}{1+x^2}\,dx$ Let $u = \tan^{-1}x$

$\frac{du}{dx} = \frac{1}{1+x^2}$; $du = \frac{dx}{1+x^2}$

$\int u^2\,du = \frac{u^3}{3} + C$ (3)

$= \frac{(\tan^{-1}x)^3}{3} + C$

(b) (i) $\int_2^{10} \frac{x\,dx}{\sqrt{x-1}}$ Let $u^2 = x-1$

$2u\frac{du}{dx} = 1$

$2u\,du = dx$; $x = u^2+1$

$\sqrt{x-1} = \sqrt{u^2} = u$

when $x = 2$, $u^2 = 2-1 = 1$ $\therefore u = 1$

when $x = 10$, $u^2 = 10-1 = 9$ $\therefore u = 3$

$\int_1^3 \frac{(u^2+1)\,2u\,du}{u} = \int_1^3 2(u^2+1)\,du$

$= 2\int_1^3 (u^2+1)\,du = 2\left[\frac{u^3}{3} + u\right]_1^3$

$= 2\left\{(9+3) - \left(\frac{1}{3}+1\right)\right\} = 21\frac{1}{3}$ (4)

(ii) $\int_1^4 \frac{dx}{\sqrt{x}(1+x)}$ Let $u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$; $\frac{dx}{\sqrt{x}} = 2du$

when $x = 1$, $u = 1$

when $x = 4$, $u = 2$

$\int_1^2 \frac{2du}{1+u^2} = 2\int_1^2 \frac{du}{1+u^2}$

$= 2\left[\tan^{-1} u\right]_1^2$ (4)

$= 2(\tan^{-1} 2 - \tan^{-1} 1)$

$= 0.644$

(iii) $\int_0^{\frac{\pi}{2}} \tan^2\left(\frac{x}{2}\right) \sec^2\left(\frac{x}{2}\right) dx$

Let $u = \tan \frac{x}{2}$

$\frac{du}{dx} = \left(\sec^2 \frac{x}{2}\right) \times \frac{1}{2}$

$\frac{du}{dx} = \frac{1}{2} \sec^2 \frac{x}{2}$

$2du = \sec^2 \frac{x}{2}\, dx$

when $x = 0$, $u = \tan 0 = 0$

when $x = \frac{\pi}{2}$, $u = \tan \frac{\pi}{4} = 1$

$\int_0^1 u^2 \times 2du = 2\int_0^1 u^2 du$

$= 2\left[\frac{u^3}{3}\right]_0^1$

$= \frac{2}{3}\left[u^3\right]_0^1 = \frac{2}{3}(1-0)$

$= \frac{2}{3}$ (4)

(c)(i) $x = 4\sin\theta$; $\sin\theta = \frac{x}{4}$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$= \sqrt{1 - \frac{x^2}{16}}$ (2)

$= \frac{\sqrt{16-x^2}}{4}$

(ii) $\int \frac{x^2 dx}{\sqrt{16-x^2}}$ Let $x = 4\sin\theta$

$\frac{dx}{d\theta} = 4\cos\theta$; $dx = 4\cos\theta\, d\theta$

$\int \frac{16\sin^2\theta\, 4\cos\theta\, d\theta}{4\cos\theta} = 16\int \sin^2\theta\, d\theta$

$= 16\int\left(\frac{1-\cos 2\theta}{2}\right) d\theta$

$= 8\int (1-\cos 2\theta)\, d\theta$

$= 8\left(\theta - \frac{\sin 2\theta}{2}\right) + C$

$= 8\theta - 4\sin 2\theta + C$

$= 8\theta - 4 \times 2\sin\theta\cos\theta + C$

$$= 8\theta - 8\sin\theta\cos\theta + C \longrightarrow ①$$

We have $\sin\theta = \frac{x}{4}$ and $\cos\theta = \frac{\sqrt{16-x^2}}{4}$

$\therefore \quad \theta = \sin^{-1}\left(\frac{x}{4}\right)$

Substitute these in ① we get

$$\int \frac{x^2\,dx}{\sqrt{16-x^2}} = 8\sin^{-1}\left(\frac{x}{4}\right) - 8 \times \frac{x}{4} \times \frac{\sqrt{16-x^2}}{4} + C$$

$$= 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{8}{16}\,x\sqrt{16-x^2} + C$$

$$= 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x}{2}\sqrt{16-x^2} + C$$

4